Income distribution and stochastic multiplicative process with reset events

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Summary. This paper examines a stochastic multiplicative process with reset events to explain the power law in the tail of personal income distribution. The tail part of the income distributions in post-war Japan persistently exhibits a power-law distribution with an exponent around 2. We find that a multiplicative process with reset events can explain this pattern. By using a default rate of corporate fundings as a hazard rate of the reset event, we obtain the correct exponent for the power-law in Japanese income.

1 Introduction

It has long been noticed since a work of V. Pareto that the tail of the income distribution obeys a power-law distribution: $\Pr(X > x) \propto x^{-\alpha}$. Recent research revealed that the tail exponent fluctuates in a certain interval over years. Using tax returns data, Souma [7] and Feenberg and Poterba [3] reported that the Pareto exponents $\alpha$ of Japan and the U.S. income distributions hover within an interval $(1.5, 2.5)$ with mean around 1. This paper gives an account for the emergence of the power law distribution and a possible cause of the Pareto exponent.

Many researchers have attempted to explain the power law in income distribution by utilizing a multiplicative stochastic process. A multiplicative process yields a log-normal distribution if the process is unconstrained. Levy and Solomon [5] showed that a power-law distribution emerges when there is a reflective lower barrier that shifts with the mean of the entire distribution. In their model, the exponent of the power-law is determined by the relative size of the lower barrier to the mean income. However, we could not find a good proxy of the lower bound that would explain the exponent in Japanese data.

An alternative way to yield a power-law distribution is a stochastic multiplicative process with reset events. Manrubia and Zanette [6] presented the power-law stationary distributions of this class of models with analysis and simulations. Gabaix [4] solved the stationary distribution of a normalized version of this process, and proposed that the variety of the exponents in income distribution may be accounted by
the dependence of the exponents on parameters of the process. We take this avenue to explain the power law in Japanese income distribution.

We view income process as follows. There are two kinds of income source: risky ones and risk free ones. Risky income source includes a part of capital and human capital. Risk free income source is typically labor and risk free assets. Risky income bears two kinds of risks: randomness in growth rates and the possibility that the income source vanishes suddenly. Hence it obeys a stochastic multiplicative process with reset events. Risk free income source serves as a lower bound of the individual income process, since most people are endowed with labor. The risk free income may vary across jobs, but the distribution has a thin tail, and the entire distribution is shifting along with mean income. If the distribution of risky income has a power-law tail, as we will show shortly, then the tail of the entire distribution will be dominated by the power-law distribution.

2 Model

In this section we formulate the process of the personal risky income. Suppose that the income $w(t)$ follows a geometric Brownian motion with “killing”. Hence we are imposing a normality of the shocks to the process. The killing rate is $\mu$. A reset point is denoted by $w_0(t)$. The reset point is also a reflective lower barrier.

\begin{align}
\frac{dw_t}{w_t} &= \gamma dt + \sigma dZ_t \quad \text{if } y_{\tau} = 0 \text{ for any } \tau \in [t, t + dt] \\
w_t &= w_t^0 \quad \text{if } y_t = 1 \\
w_t &\geq w_t^0
\end{align}

where $Z_t$ is a Wiener process and $y_t$ is a Poisson process with hazard rate $\mu$. Inequality (3) means that $w_0^t$ is the reflective lower bound. Namely, whenever $w_t < w_0^t$ happens in equation (1), $w_t$ is reset to $w_0^t$. Suppose further that the lower bound $w_0^t$ grows as fast as the drift of risky income.

\begin{align}
\frac{dw_0^t}{w_0^t} &= \gamma dt
\end{align}

The economic intuition behind this specification is as follows. Risky capital income naturally obeys a multiplicative process, as we often observe that the capital grows exponentially. By the term capital income we mean the income derived from accumulative resource broadly. Thus it includes an income from human capital (education etc.), and also the wage of managers that are often associated with the growth of the capital the managers are in charge of.

The lower bound of capital income $w_0^t$ is the income level below which the person’s consumption cannot be sustained. The bound is reflective, since the consumers can utilize their labor resource to maintain the minimum income level. This minimum income level may be random in reality, but Manrubia and Zanette [6] showed by simulation that the randomness does not alter the tail exponent of the power-law.

The growing lower bound is consistent with the empirical regularity that the ratio of labor income to capital income is roughly stable. Also, it is natural in terms of risk sharing that the capital income takes all the risk of income process and the labor income takes none when the asset holders are risk neutral and workers are risk averse.
The income growth process is "killed" by the rate $\mu$. The killing represents the evaporation of some income source. We view bankruptcy of firms or default of corporate bonds as such evaporation. Risky assets or well-paying managerial positions disappear from economy with such events. Default is an unnegligible risk of an economy. About 1% of corporate funding defaults in a typical year in Japan. We will see later that this level of default is large enough to generate a power-law distribution of income with exponent 2. The interpretation of $\mu$ as a default rate requires the default rate to be independent of the size of the firms. This is consistent with the fact that the size of firms' income and the size of defaulted firms both follow a Zipf's law ([Aoyama et.al 1] and [Axtell 2]). Also, our interpretation of $\mu$ implicitly assumes that the recipients of the risky income do not hedge the risk. It is natural, as we imagine the top tier income group as managers, entrepreneurs, and founding family of corporations rather than rentiers.

We normalize the income process $w_t$ by the lower bound $w_0^t$.

$$v_t = w_t / w_0^t$$ (5)

$$v_0 = 1$$ (6)

Then, $dv/v = dw/w - \gamma dt$ by Ito formula.

$$dv_t/v_t = \sigma dZ_t$$ if $y_t = 0$ for any $\tau \in [t, t + dt]$ (7)

$$v_t = v^0$$ if $y_t = 1$ (8)

$$v_t \geq 1$$ (9)

The process $v_t$ has a stationary distribution when $\mu > 0$. We draw on Gabaix [4] in solving the stationary distribution. First we write the Kolmogorov forward equation as:

$$\frac{\partial}{\partial t} \phi(v, t) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial v^2} v^2 \phi(v, t) - \mu \phi(v, t)$$ (10)

Suppose that a Pareto distribution $\phi = Cv^{-1-\alpha}$ solves the equation for the stationary distribution. Then $\alpha > 0$ solves the characteristic equation

$$\alpha^2 - \alpha - 2\mu/\sigma^2 = 0.$$ (11)

The positive root is

$$\alpha = \frac{1 + \sqrt{1 + 8\mu/\sigma^2}}{2}.$$ (12)

This is the Pareto exponent of the stationary distribution of $v$. The exponent carries over to the distribution of income $w$.

Numerical simulations verify this formula. Figure 1 shows the evolution of distribution of 100000 points for 2000–10000 periods where each point follows a discretized version of the $v_t$ process (7–9). It is clear that the distribution converges to a power-law distribution with exponents determined by (12).

We show that the formula (12) matches the Japanese data. We suppose that the mean of the yearly Pareto exponent is determined by a stationary exponent $\alpha$ when $\mu$ and $\sigma$ is given. We use as $\mu$ the default rate data which is quarterly announced by Bank of Japan. The default rate is the ratio of defaulted bonds over total funds of corporate sectors. The ratio fluctuates from 0.2% to 1.7% around mean 0.9705% during the period from 1970 to 1996. As for $\sigma$, we use a constant $0.1/1.022 = 0.0978$. 

for entire periods. The value is estimated from the sales growth rates of Nikkei 225 companies during 1996–2000. The growth rate has mean 1.022 and standard deviation 0.1. Then the normalized income in the model has an annual standard deviation 0.1/1.022.

Applying $\mu = 0.009705$ and $\sigma = 0.0978$ to the formula (12), we obtain a theoretical prediction $\alpha = 2.0097$. The historical mean of Pareto exponent is 2.003 for the period 1970–1996. Hence our model offers a good quantitative match with the Japanese income distribution.

The empirical time series of the default rate also exhibit an interesting correlation with the yearly Pareto exponent. Suppose that a yearly Pareto exponent is determined by (12) given the default rate of the year as $\mu$ and a constant $\sigma = 0.0978$ for the period. Figure 2 plots the predicted and actual exponents from 1970 to 1996. This theoretical prediction exhibits quite a good fit with empirical observation, as is seen in the scatter plot of the same series (Figure 3).

Figure 4 shows the same relationship as Figure 2 for a longer time series. The data for the default rate was constructed from the total value of defaulted bonds (Tokyo Shoko Research) and the total funding of corporate sectors (Ministry of Finance). The computed default rate does not exactly match with the default rate reported by Bank of Japan.

The correlation shown in Figures 2, 3, 4 cannot be taken as an evidence for that the formula (12) directly governs the year-to-year fluctuations of $\alpha$, since with our parameter values of $\mu$ and $\sigma$, it takes longer periods than our data period for the income to converge to the stationary distribution.$^3$ Nonetheless, it is possible that the default affects other parameters of the model such as the growth rate of risky income or the number of consumers in the power-law part so that it accelerates the convergence of the income distribution.

$^3$ We owe J.P. Bouchaud on this point.
**Fig. 2.** Pareto exponents of income distribution in Japan. The circle shows the actual exponent and the cross shows the predicted exponent.

**Fig. 3.** Scatter plot of the actual and implied Pareto exponents
One may think that the Pareto exponent is driven by business cycles and thus the fit obtained above merely reflects a spurious correlation. Figure 5 plots the time series of the default rate as well as the Pareto exponent and the periods of recession. It is seen that the default rate comoves with Pareto exponent, whereas recession does not correlate with either of them. This shows that the default is a non-trivial factor in determining income distributions.

The model may also explain why the exponent of the personal income distribution is volatile and large (1.5–2.5) whereas the exponent of the firm income distribution is stable and low (around 1). The important difference between a personal
income process and a firm income process is that the firm income does not have a lower bound growing as fast as the entire distribution, whereas the personal income has one because labor is endowed to most persons. If the lower bound does not grow, we obtain a modified formula for the exponent as:

\[ \alpha = \left( 1 - \frac{2\gamma}{\sigma^2} + \sqrt{\left(\frac{2\gamma}{\sigma^2} - 1\right)^2 + \frac{8\mu}{\sigma^2}} \right) / 2 \]  (13)

When the income drift is large enough so that \( \gamma > \frac{\sigma^2}{2} \), the right hand side is approximated by the first order Taylor expansion as \( \frac{\mu}{\gamma - \frac{\sigma^2}{2}} \). Thus the exponent \( \alpha \) of the firms’ income distribution is obtained when \( \mu = \gamma - \frac{\sigma^2}{2} \). Let us notice that \( \mu \) here can incorporate the growth in the number of firms in addition to the default rate. This is because an entrant firm at the lower barrier reduces the density of other income levels proportionally to the density. It is natural to think that the number of firms grows as fast as the total income of firms. Also, the magnitudes of default rate and \( \sigma^2 \) are much smaller than the growth rate of firms’ income. Hence we obtain a power-law distribution of exponent near 1 which is irresponsive to the fluctuation of the default rates.

3 Conclusion

By using a stochastic multiplicative process with reset events and lower reflective barrier, we obtain an explicit formula for the power-law exponent of income distribution. The exponent is dependent on the default rate (default rate of corporate funding) and the diffusivity of the income process. The model fits well to the historical exponent in Japanese income distribution. It is also pointed out that the default rate plays an important role to determine the year-to-year fluctuation of the exponent.

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References
